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COUNTEREXAMPLES IN MULTIVARIABLE DIFFERENTIAL CALCULUS

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Abstract

The authors present give a short review of their experience in teaching multivariable calculus and present certain type examples (counterexamples). Counterexamples considered in the paper refer to three topics of differential calculus: limits, continuity and differentiation.

Introduction

The role of examples in learning and teaching mathematics cannot be overestimated: giving a great number of examples allows students to understand the studied concepts and relations between them better, to master the methods of solving tasks and applying them to practical problems. Our experience in teaching calculus shows whether and to what degree it is helpful for students studying multivariable differential calculus to consider certain type examples (counterexamples), which illustrate a difference from the single variable case.

Unfortunately, in a lot of textbooks too much attention is paid to the idea of cross-section of multivariable functions by varying one variable independently of others at a time while keeping the others fixed. It is a good way to study multivariable functions, thus obtaining functions of one variable, but on the other hand, this leads to failure to understand the distinction between properties of multivariable functions, executed with respect to all variables, and properties, executed with respect to each variable independently.

Counterexamples considered in the paper refer to three topics of differential calculus: limits, continuity and differentiation. In spite of all examples being considered for functions of two variables in the neighbourhood of point $(0,0)$, they can also be easily adapted to the more general case of functions of more than two variables. One can find the analysis of some of the given examples in literature: $f_1, f_3, f_7, f_8, f_{12}, f_{13}, f_{14}$ (Ляшко, 1977), f_1, f_4, f_5, f_6, f_7 (Гелбаум, 1967), f_6, f_{11}, f_{21} (Бутузов, 1988).

Limits of functions of two variables

While the notions of limit and continuity look formally the same for functions of one and many variables, they are somewhat more subtle in the multivariable case. The reason for this is that we can approach a point on the line from just two directions (left or right) but in the space there is an infinite number of ways to approach a given point. It can be shown by means of examples that the value of a limit can be dependent on the choice of this direction: $\lim\limits_{\substack{x\to 0\\ y=kx}} f(x,y)$ depends on k or $\lim\limits_{\rho\to +0} f(\rho\cos\phi, \rho\sin\phi)$ depends on φ. Then the double limit does not exist. Moreover, the existence of the unique value for limits along each line through a point does not guarantee that a function has the limit. Such situation is true, for example, for function $f_1(x,y)=\dfrac{x^2 y}{x^4+y^2}$.

It is easy to see that

$$(a)\quad \lim_{\rho\to +0} f_1(\rho\cos\phi, \rho\sin\phi) = 0,$$

$$(b)\quad \lim_{\substack{x\to 0\\ y=kx}} f_1(x,y) = \lim_{\substack{y\to 0\\ x=ky}} f_1(x,y) = 0,$$

but $\lim\limits_{\substack{x\to 0\\ y=x^2}} f_1(x,y) = \dfrac{1}{2}$. It implies that double limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_1(x,y)$ does not exist. It also means that the convergence in limit (a) is not uniform with respect to $\varphi \in (0,2\pi)$ and the convergence in limits (b) is not uniform with respect to $k \in R$. The uniform convergence in limit (a) or in limits (b) would guarantee the existence of the double limit.

The iterated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exist but the double limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist for the following functions:

$$f_2(x,y) = \frac{x - y + x^2 + y^2}{x + y}, \quad f_3(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}.$$

Since $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_2(x,y) = 1$, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f_2(x,y) = -1$, by this example it is proved that an interchange of the iterated limits in general is not valid. Otherwise, the existence of the unique value for iterated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_3(x,y) = \lim\limits_{y\to 0}\lim\limits_{x\to 0} f_3(x,y) = 0$ does not guarantee that the function has the double limit.

Let us consider also examples of double limits in the case when one or both of iterated limits do not exist. For function $f_4(x,y) = x + y\sin\frac{1}{x}$ limit $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f_4(x,y)$ does not exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_4(x,y) = \lim\limits_{\substack{x\to 0\\ y\to 0}} f_4(x,y) = 0$. For function $f_5(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}$ both iterated limits do not exist, but $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_5(x,y) = 0$. By means of these examples it is proved that the existence of the double limit does not imply the existence of the iterated limits.

Continuity of functions of two variables

Continuity of multivariable functions in general cannot be proved by using single variable methods: continuity of a function at a point with respect to each variable independently is not enough to ensure continuity of this function at a point.

Function $f_6(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$ is continuous at $(0,0)$ with respect to x and with respect to y, but it is not continuous at point $(0,0)$ along the line $y = kx, k \neq 0$.

Function $f_7(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$ is continuous at $(0,0)$ along all lines $y = kx$ and $x = ky$, but it is not continuous at this point.

Differentiation of functions of two variables

The relations between partial derivatives and directional derivatives, differentiability and continuity will be discussed in this sub-section. In particular, by means of examples it can be shown that the existence of all partial derivatives at a point does not guarantee that a function is continuous or is differentiable there. It illustrates a difference from the single variable case, where differentiability implies continuity, and the term differentiable means that the derivative exists. In the multivariable case we use the much stronger sufficient condition of differentiability - continuity of all partial derivatives; however there exist differentiable functions which do not have continuous partial derivatives.

By using the following denotations:

D - function f is differentiable at point $(0,0)$,

DX (DY) - partial derivative $\frac{\partial f}{\partial x}(0,0)$ ($\frac{\partial f}{\partial y}(0,0)$) exists,

C - function f is continuous at point $(0,0)$,

CX (CY) - f is continuous at $(0,0)$ with respect to x (to y),

DC - partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$,

we construct the scheme of relations between the before-mentioned properties:

$$\mathrm{DC} \Rightarrow \mathrm{D} \Rightarrow \begin{cases} \mathrm{DX} & \Rightarrow \quad \mathrm{CX} \\ \mathrm{D}Y & \Rightarrow \quad \mathrm{C}Y \end{cases}$$

$$\mathrm{D} \Rightarrow \mathrm{C} \Rightarrow \mathrm{CX}\ \&\ \mathrm{CY}$$

The below considered functions show that none of the pointers of the scheme could be reversed vice versa (it means that none of implication signs could be replaced with a sign of equivalence):

$f_8(x,y) = \sqrt{|xy|}$, $\quad f_9(x,y) = |x| + |y|$, $\quad f_{10}(x,y) = \sqrt[3]{x^2 y^2}$,

$$f_{11}(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

$$f_{12}(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

$$f_{13}(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f_{14}(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

Table 1 illustrates the role of each function in the analysis of this scheme. For example, the location of function f_8 in row C and in column ¬D means that function f_8 is continuous at $(0,0)$, but is not differentiable.

	¬C	¬ DX , ¬DY	¬D	¬DC
C		f_9	f_8, f_9, f_{11} f_{13}	f_{10}, f_{14}
DX, DY	f_6, f_7, f_{12}		f_8, f_{11}, f_{13}	f_{10}, f_{14}
D				f_{10}, f_{14}
DC				

Table 1: Properties of the functions

It is helpful for students to analyse properties of functions f_{15}, f_{16}, f_{17}, f_{18}, f_{19} and f_{20}, which are defined by using natural parameters k, m and n. Let us note that: 1) if $m=1$, then $f_{15}=f_9$;

2) if m is even, then $f_{15}=f_{16}$ and $f_{19}=f_{20}$;

3) if $m=n$, then $f_{15}=f_{20}$ and $f_{16}=f_{19}$;

4) if k and m are even, then $f_{17}=f_{18}$.

$f_{15}(x,y) = \sqrt[m]{\lvert x\rvert^m + \lvert y\rvert^m}$	if *m* is odd	¬ DX & ¬DY (¬D)
$f_{16}(x,y) = \sqrt[m]{x^m + y^m}$	if *m* is even	¬ DX & ¬DY (¬D)
	if *m* is odd, $m \neq 1$	DX & DY & ¬D
	if $m=1$	D (DX & DY)
$f_{17}(x,y) = \sqrt[n]{x^k y^m}$ if *n* is odd	if $k + m > n$	D (DX & DY)
	if $k + m \leq n$	DX & DY & ¬D
$f_{18}(x,y) = \sqrt[n]{\lvert x\rvert^k \lvert y\rvert^m}$	if $k + m > n$	D (DX & DY)
	if $k + m \leq n$	DX & DY & ¬D
$f_{19}(x,y) = \sqrt[n]{x^m + y^m}$ if *n* is odd	if $m>n$	D (DX & DY)
	if $m<n$	¬ DX & ¬DY (¬D)
$f_{20}(x,y) = \sqrt[n]{\lvert x\rvert^m + \lvert y\rvert^m}$	if $m > n$	D (DX & DY)
	if $m < n$	¬ DX & ¬DY (¬D)

Table 2: Analysis of the properties in dependence of parameters

Mixed partials of functions of two variables

Now we consider the notion of mixed partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. It is known that the existence of continuous mixed partials at $(0,0)$ implies the equality $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0)$. Moreover, the continuity of only one from mixed partials guarantees that the last equality is true. Taking into account that for function

$$f_{21}(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0, \end{cases}$$ we have $\dfrac{\partial^2 f_{21}}{\partial x\partial y}(0,0)\neq\dfrac{\partial^2 f_{21}}{\partial y\partial x}(0,0)$,

by means of this example we can illustrate the importance of continuity for the equality of mixed partials.

The next examples show that the equality of the mixed partials at $(0,0)$ does not imply the continuity of $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at this point.

For functions

$$f_{22}(x,y)=\begin{cases} \dfrac{\sqrt[3]{x^4y^4}}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0, \end{cases}$$ and $$f_{23}(x,y)=\begin{cases} \dfrac{\sqrt[3]{x^5y^5}}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0, \end{cases}$$

the mixed partials exist and are equal at point $(0,0)$, but they are not continuous at $(0,0)$. Let us note also that function f_{22} is not differentiable at $(0,0)$, but f_{23} is differentiable at this point.

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