

# SOLUTION OF DIRECT EDDY CURRENT PROBLEMS WITH CYLINDRICAL SYMMETRY

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## ABSTRACT

*Semi-analytical solution of an eddy current problem with azimuthal symmetry is considered in the paper. The change in impedance of an excitation coil is computed in the case where the coil is located above a conducting half-space with a surface flaw in the form of a cylindrical hole. The problem is solved using the TREE method. Results of numerical calculations are presented.*

**Key words:** eddy current method, separation of variables, TREE method

## 1. Introduction

Method of integral transforms is often used in eddy current testing in order to solve axisymmetric problems for a planar multilayer medium under the assumption that the medium is infinite in two horizontal directions [1]-[3]. Analytical solutions, in general, cannot be found if the medium contains a region of finite size (a flaw) whose properties differ from the properties of the surrounding medium. In one special case where the conductivity of the flaw is close to the conductivity of the surrounding medium an approximate solution can be constructed by means of a perturbation method [4, 5]. However, the accuracy of a perturbation method cannot be directly estimated without numerical solution of the corresponding problem.

Semi-analytical solution to axisymmetric eddy current problems can be constructed by the TREE method [3]. The solutions obtained by the TREE method are not analytical since there are two steps in the procedure where numerical methods should be used. One of the steps requires to find complex roots of a transcendental equation without good initial guess for the root. An algorithm for the determination of complex eigenvalues is described in [6, 7] and is implemented in [8] for the solution of one eddy current problem. The second step of the procedure consists of numerical solution of a system of linear algebraic equations in order to determine the expansion coefficients.

One problem with a surface defect in the form of a cylindrical hole is considered in the paper. Excitation coil is assumed to be coaxial with the cylindrical hole. The system of the Maxwell's equations is solved by means of the TREE method. Results of numerical calculations are presented.



## 2. Basic equations

Suppose that a cylindrical air core coil with alternating current is located above a conducting half-space with conductivity  $\sigma$ . The half-space contains a surface flaw in the form of a cylindrical hole of radius  $c$  and height  $d$  (see Fig. 1). The coil is characterized by the following parameters:  $r_1$  and  $r_2$  are the inner and outer radii,  $z_2 - z_1$  is the height of the coil,  $N$  is the number of turns.

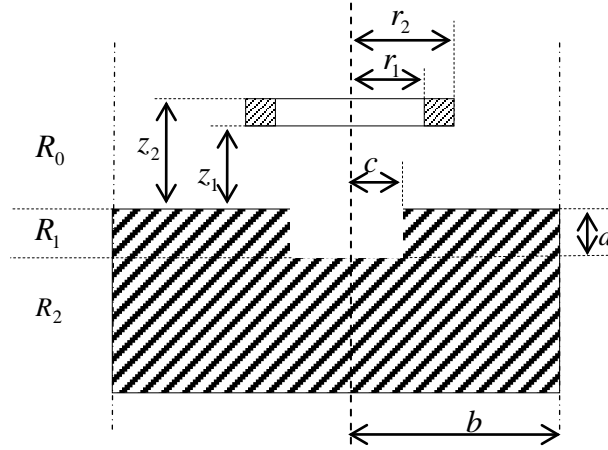


Fig. 1. A coil above a conducting half-space with a cylindrical hole.

First, we obtain the solution for the case where there is a single-turn coil of radius  $r_0$  at height  $h$  above the medium. The system of equations for the components of the vector potential in regions  $R_0 - R_2$  has the form

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I \delta(r - r_0) \delta(z - h), \quad (1)$$

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r} \frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} - j\omega\sigma_1\mu_0 A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \quad (2)$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} - j\omega\sigma\mu_0 A_2 + \frac{\partial^2 A_2}{\partial z^2} = 0, \quad (3)$$

where  $\sigma$  is the conductivity of the shaded region in Fig. 1,  $\sigma_1$  is the conductivity of region  $R_1$  ( $\sigma_1 = 0$  if  $0 \leq r < c$  and  $\sigma_1 = \sigma$  if  $c < r < b$ ),  $\omega$  is the frequency of the excitation current. The boundary conditions are

$$A_i |_{r=b} = 0, \quad i = 0, 2, \quad A_1^c |_{r=b} = 0, \quad (4)$$

$$A_0 |_{z=0} = A_1^a |_{z=0}, \quad \frac{\partial A_0}{\partial z} |_{z=0} = \frac{\partial A_1^a}{\partial z} |_{z=0}, \quad 0 \leq r < c, \quad (5)$$

$$A_0 |_{z=0} = A_1^c |_{z=0}, \quad \frac{\partial A_0}{\partial z} |_{z=0} = \frac{\partial A_1^c}{\partial z} |_{z=0}, \quad c < r < b, \quad (6)$$

$$A_1^c |_{z=-d} = A_2 |_{z=-d}, \quad \frac{\partial A_1^c}{\partial z} |_{z=-d} = \frac{\partial A_2}{\partial z} |_{z=-d}, \quad c < r < b, \quad (7)$$

$$A_1^a |_{z=-d} = A_2 |_{z=-d}, \quad \frac{\partial A_1^a}{\partial z} |_{z=-d} = \frac{\partial A_2}{\partial z} |_{z=-d}, \quad 0 \leq r < c. \quad (8)$$

The abbreviations ‘‘a’’ and ‘‘c’’ in region  $R_1$  represent air and conducting region, respectively. We also assume that the vector potential is bounded in regions  $R_0$  (as  $z \rightarrow +\infty$ ) and  $R_2$  (as  $z \rightarrow -\infty$ ).

### 3. Solution for a filamentary coil

Problem (1)-(8) is solved by the method of separation of variables (the details can be found in [3]). Omitting technical details of the derivation, we present the solution in the form

$$A_{00}(r, z) = \sum_{i=1}^n D_{1i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (h-z)} J_1(\lambda_i r), \quad (9)$$

$$A_{01}(r, z) = \sum_{i=1}^n D_{1i} e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0)}{\lambda_i J_0^2(\lambda_i b)} e^{-\lambda_i (z-h)} J_1(\lambda_i r), \quad (10)$$

$$A_1^a(r, z) = \sum_{i=1}^n J_1(p_i r) T_1(q_i c) (D_{2i} e^{p_i z} + D_{3i} e^{-p_i z}), \quad c < r < b, \quad (11)$$

$$A_1^c(r, z) = \sum_{i=1}^n J_1(q_i c) T_1(p_i r) (D_{2i} e^{p_i z} + D_{3i} e^{-p_i z}), \quad 0 \leq r < c, \quad (12)$$

$$A_2(r, z) = \sum_{i=1}^n D_{4i} e^{p_i z} J_1(\lambda_i r), \quad (13)$$

where

$$T_1(q_i r) = J_1(q_i r) Y_1(q_i b) - J_1(q_i b) Y_1(q_i r),$$

$$\lambda_i = \frac{\alpha_i}{b}, \quad \alpha_i \text{ are the roots of the equation } J_1(\alpha) = 0, \quad p_i = \sqrt{q_i^2 + j\omega\sigma\mu_0}, \quad p_{1i} = \sqrt{\lambda_i^2 + j\omega\sigma\mu_0},$$

$p_i$  are the complex roots of the equation

$$q_i T_1'(q_i c) J_1(p_i c) = p_i J_1'(p_i c) T_1(q_i c), \quad (14)$$

and notations  $A_{00}(r, z), A_{01}(r, z)$  correspond to regions  $0 < z < h$  and  $z > h$ , respectively. The first term on the right-hand sides of (9) and (10) is the induced vector potential:

$$A_0^{ind}(r, z) = \sum_{i=1}^n D_{1i} e^{-\lambda_i z} J_1(\lambda_i r). \quad (15)$$

The coefficients  $D_{2i}$  and  $D_{3i}$  satisfy the following system of algebraic equations

$$\sum_{i=1}^n [(\lambda_k + p_i) a_{ki} D_{2i} + (\lambda_k - p_i) a_{ki} D_{3i}] = \mu_0 I r_0 J_1(\lambda_k r_0) e^{-\lambda_k h}, \quad (16)$$

$$\sum_{i=1}^n [(p_{1k} - p_i) a_{ki} e^{-p_i d} D_{3i} + (p_{1k} + p_i) a_{ki} e^{p_i d} D_{4i}] = 0, \quad (17)$$

$$k = 1, 2, \dots, n,$$

$$a_{ik} = T_1(q_i c) \tilde{a}_{ik} + J_1(p_i c) \tilde{\tilde{a}}_{ik},$$

$$\tilde{a}_{ik} = \frac{c}{\lambda_k^2 - p_i^2} (\lambda_k J_2(\lambda_k c) J_1(p_i c) - p_i J_1(\lambda_k c) J_2(p_i c)),$$

$$\tilde{\tilde{a}}_{ik} = \frac{1}{\lambda_k^2 - q_i^2} \left\{ c q_i J_1(\lambda_k c) [J_2(q_i c) Y_1(q_i b) - J_1(q_i b) Y_2(q_i c)] + \right. \\ \left. + c \lambda_k J_2(\lambda_k c) [J_1(q_i b) Y_1(q_i c) - J_1(q_i c) Y_1(q_i b)] \right\}.$$

It can be shown that the coefficients  $D_{1i}$  can be found using the formula

$$D_{2k} = \frac{2}{b^2 J_0^2(\lambda_k b)} \sum_{i=1}^n a_{ki} (D_{2i} + D_{3i}) - \frac{\mu_0 I r_0 J_1(\lambda_k r_0) e^{-\lambda_k h}}{\lambda_k b^2 J_0^2(\lambda_k b)}. \quad (18)$$

### 4. Solution for a coil with finite dimensions

Using the formula for the induced vector potential (15) one can obtain the induced vector potential due to currents in the whole coil shown in Fig. 1:

$$A_{0coil}^{ind}(r, z) = \int_{r_1}^{r_2} \int_{z_1}^{z_2} A_0^{ind}(r, z, r_0, h) dr_0 dh. \quad (19)$$

The change in impedance of the coil is found by the formula

$$Z^{ind} = \frac{2\pi j \omega}{I} \frac{N}{(r_2 - r_1)(z_2 - z_1)} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r A_{0coil}^{ind}(r, z) dr dz. \quad (20)$$

Substituting (15) into (19) and (20) we obtain the change in impedance of the coil

$$Z^{ind} = \frac{2j\omega\pi\mu_0 N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2} \sum_{i=1}^n \frac{(e^{-\lambda_j z_1} - e^{-\lambda_j z_2})^{\lambda_j r_2}}{\lambda_j^3} \int_{\lambda_j r_1}^{\lambda_j r_2} \xi J_1(\xi) d\xi \sum_{k=1}^n Y_{ik} \frac{(e^{-\lambda_k z_1} - e^{-\lambda_k z_2})^{\lambda_k r_2}}{\lambda_k^3} \int_{\lambda_k r_1}^{\lambda_k r_2} \xi J_1(\xi) d\xi, \quad (21)$$

where the elements of the matrix  $Y$  are bulky and are not shown here for brevity.

Using formula (21) we calculated the change in impedance of the coil for the following parameters of the problem:  $\mu_0 = 4 \cdot 10^{-7} \pi$ ,  $\sigma = 3 \text{ Ms/m}$ ,  $c = 2.2 \text{ mm}$ ,  $r_2 = 5.5 \text{ mm}$ ,  $r_1 = 3 \text{ mm}$ ,  $z_1 = 0.2 \text{ mm}$ ,  $z_2 = 3 \text{ mm}$ ,  $d = 0.9 \text{ mm}$ ,  $N = 100$ . The results are shown in Fig. 2 for six different frequencies from 2 kHz to 7 kHz.

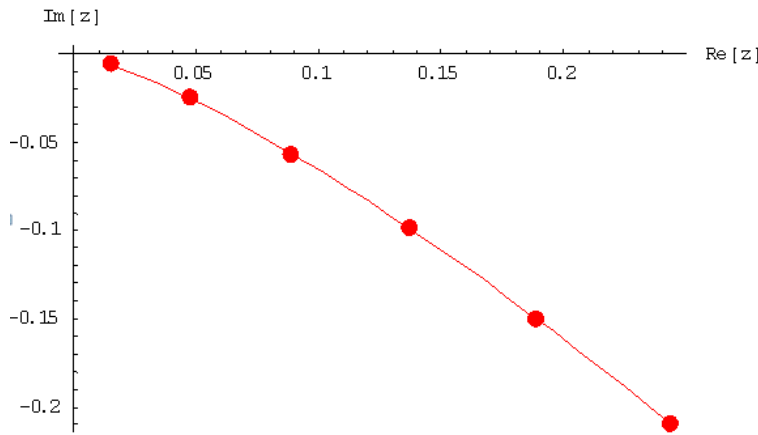


Fig. 2. The change in impedance of the coil for six different frequencies.

## 5. Conclusions

Semi-analytical solution of eddy current problem is presented in the paper. A coil with alternating current is located above a conducting half-space with a flaw in the form of a conducting cylinder whose axis coincides with the axis of the coil. The solution is found by means of the TREE method. Results of numerical computations are presented.

## 5. Acknowledgement

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## 6. References

- [1] Tegopoulos J.A., Kriezis E.E.: Eddy currents in linear conducting media, Elsevier, New York, 1985.
- [2] Antimirov M.Ya., Kolyshkin A.A., Vaillancourt R.: Mathematical models for eddy current testing, CRM, Montreal, 1997.
- [3] Theodoulidis T.P., Kriezis E.E.: Eddy current canonical problems (with applications to nondestructive evaluation), Tech Science, New York, 2006.
- [4] Antimirov M.Ya., Kolyshkin A.A., Vaillancourt R.: Eddy current non-destructive testing by a perturbation method, *Journal of non-destructive evaluation*, Vol. 10, 1991, 31–37.
- [5] Antimirov M.Ya., Kolyshkin A.A., Vaillancourt R.: Application of a perturbation method to the solution of eddy current testing problems, *IEEE Transaction on Magnetism*, Vol. 30, No. 3, 1994, 1247–1252.
- [6] Lyness J.N.: Numerical algorithms based on the theory of complex variable, *Proc. ACM*, 1967, 125 – 133.
- [7] Delves L.M., Lyness J.N.: A numerical method for locating the zeros of an analytic function, *Mathematics of Computation*, Vol. 21, 1967, 543 – 560.
- [8] Theodoulidis T.P., Bowler J.R. Interaction of an eddy-current coil with a right-angled conductive wedge, *IEEE Transactions on Magnetism*, Vol. 46, No. 4, 2010, 1034 – 1042.