Evaluation Of Dynamics Of The Vix Index Via Heston Model

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Abstract. A methodology for the estimation of parameter of a stochastic model using discontinuous models (ARIMA class) and based on the financial market data is introduced. This approach helps to simplify financial derivative pricing problems under various underlying stochastic processes. We show how to apply our technique to the financial index VIX - a market mechanism that measures the 30-day forward implied volatility of the underlying index, the S&P 500. Also the results with regression model of time series which produced by Heston volatility model are considered.

Key words: diffusion processes, time series, VIX index, CIR model, Heston model, ARIMA, ARCH.

1. Introduction

The purpose of this paper is to give a reader an overview of the ARIMA techniques which could be useful in the case of the continuous Markov process representation in discrete time. As we know, every continuous Markov process can be considered as a limiting case of a discontinuous Markov process and that the solutions of the Kolmogorov diffusion equations can be approximated by solutions of the Kolmogorov differential equations [1].

For further discussion, let take a look to the Heston volatility model. This approach is adopted to give the reader an intuitive understanding of the Heston model, rather than an overly technical one, so that the sections that follow are easily absorbed. If further technical details are desired, the reader is directed to the relevant references.

(Heston 1993) proposed the following the model:

$$dS_t = \mu S_t dt + \sqrt{V_t}\, S_t dW_t^1, \qquad (1)$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\, dW_t^2, \qquad (2)$$

$$dW_t^1 dW_t^2 = \rho dt,$$

where $\{S_t\}_{t>0}$ and $\{V_t\}_{t>0}$ are the price and volatility processes, respectively, and $\{W_t^1\}_{t>0}$, $\{W_t^2\}_{t>0}$ are correlated Brownian motion processes (with correlation parameter ρ), μ is deterministic risk free interest rate. $\{V_t\}_{t>0}$ is a square root mean reverting process, first used by (Cox, Ingersoll & Ross 1985 [3]), with long-run mean θ, σ standard deviation and rate of reversion k. ρ is referred to as the volatility of diffusion. All the parameters $k, \mu, \theta, \rho, \sigma$ are time and state homogenous.

ρ , which can be interpreted as the correlation between the log-returns and volatility of the asset, affects the heaviness of the tails. ρ > 0, then volatility will increase as the asset price/return increases. This will spread the right tail and squeeze the left tail of the distribution creating a fat right-tailed distribution.

Conversely, if ρ <0, then volatility will increase when the asset price/return decreases, thus spreading the left tail and squeezing the right tail of the distribution creating a fat left-tailed distribution (emphasizing the fact that in most cases equity returns and its related volatility are negatively correlated). Besides, ρ affects of the skew of the distribution.

There are many economic, empirical, and mathematical reasons for choosing a model with such a form (see [2] for a detailed statistical/ empirical analysis). Empirical studies have shown that an asset's log-return distribution is non-Gaussian. It is characterised by heavy tails and high peaks (leptokurtic). There is also empirical evidence and economic arguments that suggest that equity returns and implied volatility are negatively correlated (also termed 'the leverage effect'). This departure from normality plagues the Black-Scholes-Merton model with many problems. In contrast, Heston's model can imply a number of different distributions.
A contingent claim is dependent on one or more *tradable* assets in the Black-Scholes-Merton model. The randomness in the option value is solely due to the randomness of these assets. Since the assets are tradable, the option can be hedged by continuously trading the underlying. This makes the market complete, i.e., every contingent claim can be replicated.

In a stochastic volatility model, a contingent claim is dependent on the randomness of the asset $\{S_t\}_{t>0}$ and the randomness associated with the volatility of the asset's return $\{V_t\}_{t>0}$. Only one of these is tradable. Volatility is not a traded asset. This renders the market incomplete and has many implications to the pricing of options and other financial instruments [4].

Therefore we have to find a discrete representation of the Heston model and find possible ARIMA class model to show the hypothesis existence about residuals heteroscedastity using time series technique. Otherwise, we can't use Heston model for the chosen financial instrument pricing.

2. Heston model in discrete time

2.1. Discretizing of the price process

Firstly, we must find the solution for the equation (1) and make the discrete representation of this stochastic model. The following diffusion process postulated in the (1) model:

$$dS_t = \mu S_t dt + \sqrt{V_t}\, S_t dW_t \ .$$

Let's make substitution

$$y_t = \ln S_t \quad \text{and consider } V_t = const$$

The next step is to apply Ito's theorem. In particular,

$$dy_t = (\ln(S_t))' \mu S_t dt + (\ln(S_t))' \sqrt{V_t}\, S_t dW_t \ + \frac{1}{2}(\ln(S_t))'' V_t^2 \mu S_t^2 dt =$$

$$= \mu dt + \sqrt{V}\, dW_t \ - \frac{1}{2} V^2 \mu dt =$$

$$= (\mu - \frac{1}{2} V^2) dt + \sqrt{V}\, dW_t \ . \qquad (3)$$

Solving equation (3) we get:

$$y_t - y_0 = \int_0^t (\mu - \frac{1}{2}V^2)dt + \int_0^t \sqrt{V}\, dW_s =$$

$$= (\mu - \frac{1}{2}V^2)dt + \sqrt{V}\, dW_t \ .$$

Or

$$\ln S_t = (\mu - \frac{1}{2}V^2)dt + \sqrt{V}\, dW_t \ .$$

Then

$$S_t = S_0 \exp((\mu - \frac{1}{2}V^2)t + \sqrt{V}\, W_t \ .$$

In a sense, we get option (or other financial instrument) price equation which depends from previous price level.

But, it is easy to deal with substitution $y_t = \ln S_t$ and get a recursive expression for y_t in terms of its previous value. Firstly, we evenly subdivide the interval [*0; T*] into N subinterval and let $t_i = i\frac{T}{N}$ for $t_i = i,...,n$. In addition, we denote each time-step as $\Delta t = t_i - t_{i-1}$ (in practice we can take $\Delta t = 1$) and $\mathrm{W_t} \sim \varepsilon_\mathrm{t}$, where $\varepsilon_\mathrm{t} \sim \mathrm{N}(0,1)$.

In general, we have

$$y_t - y_{t-1} = (\mu - \frac{1}{2}V^2) + \sqrt{V}\, \varepsilon_t . \qquad (4)$$

After evaluation of the equation (4) we can see that continuous Markov process could be represented in discrete time. The main part of the Heston model now is ARIMA (AR(1)) process. Nowadays, AR(1) process is quite enough studied and for further model building or testing purposes we can use ARIMA technique to reject or to accept model consistency with observable financial instrument's price data.

2.2 Discretizing of the volatility process

For the second part of Heston model (equation (2)) we can use the following approach. While a variety of alternatives exist for discretizing the stochastic differential equations that we are dealing with, in the case of the CIR model we can actually solve for the diffusion process explicitly [5]. To see how this is done, consider the following diffusion process postulated in the CIR model:

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\, dW_t \ . \qquad (5)$$

We now strip out the drift term and denote it as

$$Y_t = k(\theta - V_t)dt \qquad (6)$$

The next step is a bit odd, but necessary, and involves pre-multiplying equation (6) by e^{kt} . This operation yields

$$e^{kt}Y_t = f(V_t) = e^{kt}(k(\theta - V_t) \qquad (7)$$

This is, in fact, the trick in the derivation. We have a function that depends on a stochastic process, $\{V_t\}_{t>0}$.
We would like to describe its differential dynamics, and to do so we may apply Ito's theorem. Before we do this, let us compute the required partial derivatives:

$$\frac{\partial f(V_t)}{\partial t} = ke^{kt}Y_t;$$
$$\frac{\partial f(V_t)}{\partial V_t} = -ke^{kt};$$
$$\frac{\partial f(V_t)}{\partial V_t^2} = 0.$$

We now have everything that we need to apply Ito's theorem. In particular,

$$f(V_t) - f(V_0) = \int_0^t \frac{\partial f(V_t)}{\partial t} ds + \int_0^t \frac{\partial f(V_t)}{\partial V_t} dV_s + \frac{1}{2}\int_0^t \frac{\partial f(V_t)}{\partial V_t^2} ds =$$
$$= \int_0^t ke^{ks} Y_s ds - \int_0^t ke^{ks}[Y_s ds + \sigma dW_s] =$$
$$= -\int_0^t ke^{ks}\sigma dW_s. \qquad (8)$$

Inspection of equation (8) reveals that we have a recursive expression for V_t in terms of its previous value, V_0. A bit of manipulation will make this clearer:

$$e^{kt}Y_t - e^{kt}Y_0 = -\int_0^t ke^{ks}\sigma dW_s;$$
$$ke^{kt}(\theta - V_t - k(\theta - V_0)) = -\int_0^t ke^{ks}\sigma dW_s;$$
$$-ke^{kt}V_t = -k\theta e^{kt} + k\theta - kV_0 - \int_0^t ke^{ks}\sigma dW_s;$$
$$V_t = \theta(1 - e^{kt}) + e^{kt}V_0 + \int_0^t ke^{ks}\sigma dW_s. \qquad (9)$$

In a sense, we are finished, as we have a recursive expression for V_t in terms of its previous value. Firstly, we evenly subdivide the interval [*0; T*] into *N* subinterval and let $t_i = i\frac{T}{N}$ for $t_i = i,\ldots,n$. In addition, we denote each time-step as $\Delta t = t_i - t_{i-1}$.
In general, we have

$$V_{t_i} = \theta(1 - e^{-k\Delta t}) + e^{-k\Delta t}V_{t_{i-1}} + \varepsilon_{t_i}, \qquad (10)$$

where

$$\varepsilon_{t_i} = \int_{t_{i-1}}^{t_i} ke^{-k(t_i - s)}\sigma dW_s.$$

In other words, we have the first two moments of the Gaussian transition density of V_t . Specifically,

$$V_{t_i} / F_{t_{i-1}} \sim \mathrm{N}(\theta(1-e^{kt}) + e^{kt} V_{t_{i-1}}, \varepsilon_{t_i}^2).$$

All that remains, to get this expression into a form that can aid us in our simulations, is to find a more convenient way to express ε_{t_i} . In fact, it might not yet be obvious that ε_{t_i} is actually the variance of our transition density. This is true by virtue of the fact that ε_{t_i} is a stochastic integral and, as such, it has a zero expectation. We also recall that the quadratic variation process of the Brownian motion (i.e., W_t)
is t. In particular, this means that

$$E[\varepsilon_{t_i} / F_{t_{i-1}}] = 0.$$

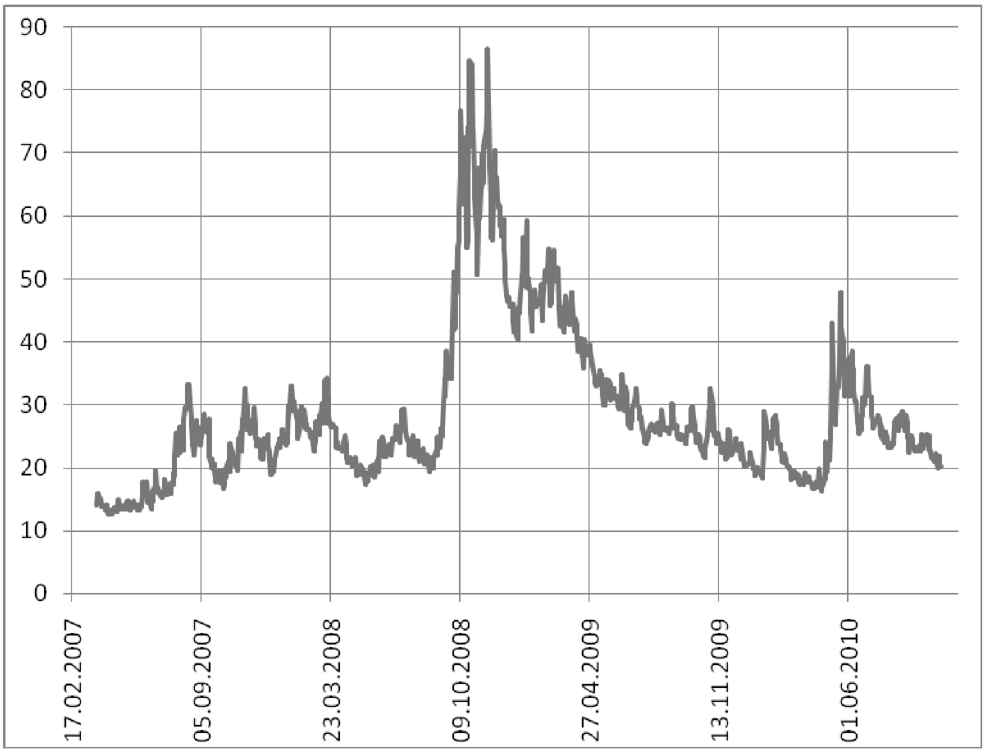
With the CIR model, however, the transition density follows a non-central χ^2-quared distribution, which is rather difficult to handle. Fortunately, Ball and Torous (1996) show that, over small time intervals, diffusions arising from stochastic differential equations behave like Brownian motion and, thus, to assume a normal transition density is probably a good approximation. Thus, for the purposes of simulation we can use the first two moments of the non-central χ^2-squared distribution and assume that

$$V_{t_i} / F_{t_{i-1}} \sim \mathrm{N}[\theta(1-e^{kt}) + e^{kt} V_{t_{i-1}}, \frac{\theta\sigma^2}{2k}(1-e^{kt})^2 + \frac{\sigma^2}{k}(e^{-kt} - e^{-2kt}) V_{t_{i-1}}].$$

Taking into account, that studying financial instruments which have χ^2-quared distributions could take a lot of time and lead to unpredictable results, the situation could be improved if we used regressions (autoregressions).
This fact allows us to capture achieved results with regression model of time series which produced by Heston volatility model.

3. **Time series estimation based on ARMA models**

Let consider one example of combine of previous methods and ARIMA techniques. We’ll analyze the VIX - **Market Volatility Index –** daily data from 27.03.2007. to 21.10.2010. The VIX is a market mechanism that measures the 30-day forward implied volatility of the underlying index, the S&P 500. Being able to meaningfully interpret movements in the VIX and its reaction to market events can give investors an edge in managing the risk and profitability of their trading book and in designing portfolio strategies using VIX derivatives to capitalize on the dynamic and time-varying correlation of the VIX with its underlying S&P 500 Index.



Graph 1 . **Market Volatility Index –** daily data from 27.03.2007. to 21.10.2010.

One can see from the graph 1 that the series seem to be nonstationary, especially in short time consideration. Besides, we observe some change of tendencies at the begin of October, 2008. The correlogram of the whole taken series shows us that autocorrelations decay slowly, this is typical for nonstationary series. And the augmented Dickey-Fuller test states presence of the unit root. So in present form whole data from 27.03.2007 to 21.10.2010 cannot be used for the stationary model construction and forecasting.
We can analyse first differences of our data. These new series DVIX seem to be conditionally heteroscedastic. They have not got any unit root, as shows Dickey-Fuller test. And the correlogram hasn't slowly decayed autocorrelations. We have chosen an appropriate model AR(1) with ARCH(2) residuals. But the stability test (Chow Breakpoint Test) certifies changing of the model about October, 2008.
So we decided to analyse the data after October, 2008 and tried to do it for the initial series. Series have clearly defined trend in this period.

So we exclude the time trend and analyse new series vix2 during the period from 10.10.2008 to 21.10.2010:

$$vix2_t = vix_t - 0.0450460613 \cdot t.$$

The system of equations (4),(13) which was received by discretezation of Heston model approximately corresponds to the AR(1) model with GARCH-M(1,0) residuals, where the squared variance with the fixed coefficient 0.5 is included in AR(1) equation for $vix2_t$.

We are successfully estimated this model for time series $vix2_t$ with the programm WINRATS and received the system in the following form:

$$\begin{cases} vix2_t = -16.743 + 0.5\,V_t^2 + 0.954 \cdot vix2_{t-1} + \varepsilon_t \\ \varepsilon_t = \nu_t \sqrt{V_t} \\ V_t = 5.9934 + 0.04138979 \cdot V_{t-1} \end{cases}.$$

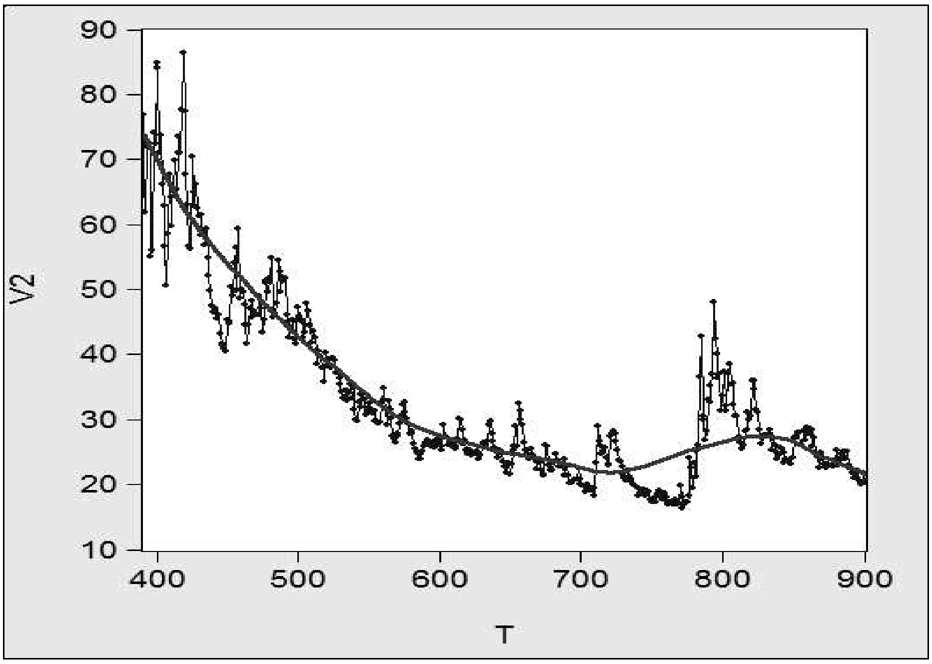
Moreover, the coefficient at V_t^2 was fixed, i.e. it is equaled by 0.5. Unfortunately, we couldn't assert that this is the best model for our data. At first, the model is appeared sensible to initial data. Secondly, if we decided to give up fixing of the coefficient at V_t^2, the actual model will appear a little different, and other coefficients will change .

$$\begin{cases} vix\,2_t = -10.3265 + 0.24347\ V_t^{\,2} + 0.95319\ \cdot vix\,2_{t-1} + \varepsilon_t \\ \varepsilon_t = v_t \sqrt{V_t} \\ V_t = 6.7326 + 0.0849\ \cdot V_{t-1} \end{cases}.$$

We can assume that some other ARMA / GARCH model will be better than this one. We have considered many similar models and the above-mentioned model indeed appears the most suitable model for our data.

Thus, the regression analysis gives us more possibilities for the analysis of such data, but the model of Heston allows us to narrow down the scopes of search of suitable model, that conversely is a pretty labor intensive process.

Graphically the estimation problem can be also successfully decides in the package of E-views by the method of local regression by the direct choice of suitable kernel and evaluation of standard deviation of realization of initial row from a hypothetical curve (this curve is represented as a red color curve in the picture).



Graph 2. The time series $vix2_t$ estimation with method of local regression

Conclusions

A discrete representation of Heston stochastic model was applied to solve the problem of stochastic model consistency with financial time series data. Model discretization shows that evaluation of parameters using ARIMA technique is quite easier. However, VIX data simulation as a result forward option price evaluation via Heston model is not the best idea. That is why above mentioned approach can be used for different class of diffusion models as a quick test of the model consistency.

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